All the Math We Cannot See

Alessandro Della Corte

Doing nothing is as good as collaborating.

A. Doerr, *All the light we cannot see*

Years ago, a friend who had recently decided to become a nun told me she wanted to enter the cloister. "Why?" I asked, surprised. "It's important; the cloistered sisters support the entire Church with their prayer." It was not, therefore, as I had imagined out of pure ignorance, a personal work of purification or introspection, but an endeavour aimed at the good of the world out there. And she believed it: nuns in silent prayer, each in her own cell, make an important contribution to the Good.

But, in the past as today, few young people choose vows of cloister. The majority of young people want a social life and, secondarily, a job, and university is usually a way to pursue both goals. A minority of students choose STEM degree programs, a minority of these choose mathematics, and a minority of the latter choose a curriculum that prioritizes pure mathematics over applied or teaching-oriented alternatives. From the perspective of the general public, this residual portion of students might seem hardly relevant, but there are actually good reasons to believe the opposite.

Two of these reasons are well known, as they are regularly invoked in popular science and scientific journalism. According to the first argument, which we will call "the maker's," abstract mathematical results, seemingly disconnected from reality, have often proven fertile for significant practical developments, providing the basis for new technology. The second argument, which we will call "the thinker's," is more intrinsic: the mathematical universe is characterized by such richness, beauty, and depth that it makes it an object of study worthy in itself. This study puts us in contact with a level of existence (how much conventional and how much objective I won't try to discriminate here) inaccessible in other ways, and any reformulation of it in purely applied terms would constitute an irreparable misrepresentation.

The maker's argument is simply true, the thinker's is certainly defensible. Both, however, are compatible with the idea that the vast majority of those who professionally engage in pure mathematics make no contribution to the general interest. Despite loud propaganda that often claims the contrary, indeed, only a tiny portion of theoretical results truly has technological fallout, and only an even more select elite, consisting of theorems of exceptional beauty and unexpected revelatory power, will be

studied in the decades or centuries to come as part of the mathematical culture worthy of being passed down.

This romanticized vision, which exalts the role of the individual, is consistent with the research funding policy pursued in recent decades by the European Union: not "a little to everyone," but "everything to very few," based on sophisticated but adventurous predictions about which lines of research will prove most promising. Naturally, universities have adapted to this narrative, choosing to advertise their mathematics degree programs by winking now at the maker, now at the thinker, with the result of often appearing not very credible to potential enrolees, who are young but not necessarily naive. In any case, the image that emerges through this way of explaining the social utility of pure mathematics is quite depressing: so many little hamsters spinning wheels that are almost all useless, most of whom are destined for irrelevance and oblivion, while a very few lucky ones stumble upon some rare gem.

I believe that, fortunately, this alienating picture is far from reality, and in particular, I think there are important reasons, of social ecology even before philosophical or utilitarian ones, why it would be healthy to preserve, and possibly strengthen, a widespread community of capable scholars who engage in pure mathematics. For, in fact, before producing results, pure mathematics produces a certain kind of practices and habits, and thus a certain kind of people. This group is certainly not indistinct, but is broadly homogeneous internally and clearly differentiated externally in terms of certain important methodological characteristics. So let's take a closer look at the ecosystem we are talking about.

As with researchers in all other fields of human knowledge, the goal of those who do pure mathematics is to develop new and interesting results, where the second adjective can naturally be understood in many senses. To achieve this goal, a long apprenticeship is usually necessary. Typically, it goes like this: a student becomes passionate about a topic, chooses it as the subject for their thesis, and then tries, if the interest continues, to obtain a PhD scholarship to spend a longer period, usually three to five years, learning things, meeting people, and finally producing autonomous and significant results. At that point, through an obstacle course whose rules change continuously, they will continue to study the things they like, if they want and are lucky, for their entire working life, often broadening their field of interests. So far, nothing different from those who work in other so-called "hard sciences."

On the other hand, there are many and quite distinct peculiarities. One, fundamental, is rarely perceived from the outside: the problems that pure mathematicians deal with are, as a rule, studied by very small communities of specialists. A typical PhD thesis, for example, may be easily understandable to only a few hundred or even a few dozen people in the world; among the humanities, an effective parallel seems to me that with philology: a critical edition of a medieval Latin text is unlikely to be readable outside a restricted group of experts. In the mathematical community, this limitation of the target audience for a research work is not at all seen as a symptom of irrelevance. It is simply the consequence of the fact that the theories are deep and layered, the imaginable questions are virtually

infinite, and the technical preparation necessary to address them seriously is difficult to replicate in many different directions (exceptions exist, of course). The social life of pure mathematicians, therefore, is made up of small communities that meet periodically at international conferences and in which reputation within one's circle of specialists often counts more than the CV or the list of publications. This subdivision into small communities determines important methodological consequences. Each community, when using results developed outside its specific area of expertise, must be able to trust, with a wide margin of safety, what others have produced, because any subsequent development that will use result X as an intermediate step will depend, forever, on the correctness of X. This mechanism works, on a historical scale, only because of a methodological characteristic that pure mathematics has pursued since the time of Euclid: the search for "absolute" rigor. In the proof of a theorem, there can be no step that, in principle, is not completely correct from a logical point of view. If, for example, proving result X requires the separate resolution of 1000 complicated particular cases, and 999 of them have been successfully addressed while only one resists, mathematicians do not consider X "practically proven"; they consider it, as a rule, simply an open problem still, and they will never make a subsequent proof depend on the correctness of X if the thousandth case has not yet been settled. The mathematical community has developed a constructive awareness, which translates into an appropriate code of individual behaviour, toward the delicacy of this collective task that often surprises other scholars. When an error is identified (which, as is obvious, happens regularly), the authors themselves are usually the first to have an interest in making it known, and they are generally grateful if errors they have committed are pointed out to them by others. Obviously, this is not a superior "intrinsic honesty" of mathematicians: simply, for the reasons mentioned above, the community has seen its own peculiar sensitivity evolve, which punishes honest errors lightly but severely punishes, for example, making others waste time by omitting or delaying the reporting of identified mistakes.

The rigorously deductive structure of pure mathematics is a methodological characteristic that defines it and runs through all its areas, which constitutes a powerful counterweight to the highly specialized character described above. There are no cultural barriers that prevent a differential geometer from reading a number theory article, as there might be between an expert in 20th-century history and one in the Upper Paleolithic. Obviously different, compared to a specialist, are the time and patience that must be invested in reading and the sensitivity with which one will be able to judge the relevance of the novelties presented.

Another difference, quantitative, concerns the net output compared to other hard sciences. In mathematics, the volume of results produced is not as large as elsewhere, and the battle of *publish or perish* is fought at a pace that recalls artisanal work more than industrial. Just take a look at researchers' bibliometric indices and the impact factors of journals to realize that a pure mathematician produces on average many fewer articles than, for example, theoretical physicists or computer scientists. Each article has on average fewer authors and is, as a rule, the culmination of a long work that will then require another long effort on the part of editors and reviewers to be critically examined before becoming part of shared knowledge through one of the main journals.

The style in which articles are written reflects these peculiarities. Naturally, there are variable characteristics from one work to another, but on average, pure mathematics articles feel little need to rhetorically signal the importance and novelty of their content. "*We propose a radically novel approach that may overcome the crucial difficulties in this fascinating challenge*": a tone like this, familiar in other fields, is almost never found in important mathematical articles. For example, the Abstract of G.J. Perelman's article in which decisive steps are taken to solve one of the most famous problems that have ever existed, the Poincaré conjecture, simply announces the technical result: "*We present a monotonic expression for the Ricci flow, valid in all dimensions and without curvature assumptions*." Any expert would immediately understand that it was an important result, and it was only to them that the author was speaking.

The gatekeeping of the community is extreme, but it takes different forms compared to other fields. The purely technical level required to access publication in an important journal is very high, as it is elsewhere, but in pure mathematics, the leap in terms of selectivity and seriousness of the editorial process between the (perhaps a hundred or so) truly prestigious journals and the (thousands) of decent/mediocre/weak journals is particularly wide. Sloppy notation or convoluted exposition can be sufficient to receive a desk-rejection (an immediate rejection, without in-depth analysis of the article's content) even if the announced results were potentially relevant. On the other hand, the funding required for research is infinitesimal compared to experimental sciences, so the probability that an unknown, typically a young person, makes a contribution at the highest level, and that this contribution is recognized by the community, is higher than in other fields. In this specific sense, gatekeeping is less stringent than elsewhere.

Finally, while often the most important researchers in other sectors have no particular interest in teaching, pure mathematicians generally love to teach (to other mathematicians; less in service courses). This is explained in good measure by the sincere and disinterested enthusiasm that most of them have toward their discipline, one of whose effects is the desire to share. But there are also more functional reasons: for example, given the very strong interconnection of mathematics itself, rethinking elementary aspects, necessary when preparing a course, often helps to produce new ideas. Summing up, there thus exists a community: i) fragmented into numerous constellations each in turn subdivided down to micro-tribes of specialists often made up of a few dozen people; ii) gruff toward those who do not conform in methods, notation, expository style, level of rigor, knowledge of results already obtained; iii) sober and measured in manners but extremely competitive in substance; iv) generally honest and open in recognizing the value of novelties; v) dedicated to difficult problems that are hard to explain in a non-technical way (the parallel with philology comes to mind again); vi) typically genuine, even childlike in their love for their work, and well-disposed toward teaching.

What does the world out there make of all this? In particular, what does it make of those thousands of

scholars who will never produce a result with revolutionary technological or conceptual fallout? I believe that much of the social utility of this community lies in the fact that it performs work that, in terms of method and the social dynamics it promotes, is transversal to the more common trajectories outside of it. Slowness, caution, attention to detail, disinterest in attention from a large public, annoyance toward rhetorical tools, healthy awareness of the smallness of one's contribution: characteristics that elsewhere are mostly penalized by dominant social mechanisms, here are constitutive. Inevitably, the different communities and sensitivities come into contact, through their products, certainly, but also directly, through exchange between people, institutionalized (lectures, multidisciplinary conferences, meetings of academic bodies, research projects) or informal (from impromptu collaborations to the simple coffee machine break). In all its interactions with the "world," while pursuing an instinctive principle of self-preservation that tends to emphasize differences and boundaries, the community of pure mathematicians exercises a silent and calm influence on other fields, and also on the work of people not necessarily involved in the world of research: technicians of all kinds, communication professionals, artists, and, obviously and directly, teachers.

Beyond the cultural or technological value of the results, beyond the interest of the thinker and the maker, there exists a collective interest in keeping pure mathematics functioning as a widespread social phenomenon. Obviously, this has been happening without interruption for several centuries, so it is not possible to provide direct counterfactual proof. We cannot say, empirically, what kind of university (or society) would remain if this micro-environment shrunk until it disappeared. However, I believe that its simple survival, in good health, through very different ages, driven by interests often far from the pursuit of scientific development per se, argues in favour of the importance of its social role, and it serves as a quiet, enduring counter-narrative to market logic. But as always, in the absence of direct empirical evidence, it takes an act of faith, if not a religious one, then a civic one. The Church performs it when it believes that the Good in the world is supported by the silent prayer work of cloistered nuns. I hope not to be judged heretical if I believe that the Good is also supported by the silent work through which someone somewhere produces all the math we cannot see.